Effective Local Volatility Model – with Application to Pricing American Basket Options

QFRG Seminar

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Outline

- Introduction, motivation and inspiration
- Model overview
- Numerical examples
- Concluding thoughts

The problem: how to price a basket option (efficiently)?

- Consider a collection of N stocks $S_1(t), S_2(t), \ldots, S_N(t)$ with prices driven by risk-neutral processes given by

$$\mathrm{d}S_i(t) = rS_i(t)\mathrm{d}t + \psi_i(\cdot)\mathrm{d}W_i(t), \quad \mathrm{d}W_i(t)\mathrm{d}W_j(t) = \rho_{i,j}\mathrm{d}t$$

where $\psi(\cdot)$ is some, possibly stochastic, volatility function.

- Let $B(t)$ be the price of a basket made up of ω_n shares of each stock:

$$B(t) = \sum_{j=1}^{N} \omega_j S_j(t), \quad \omega_j \in \mathbb{R}^+,$$

- The t_0 price of a European call on $B(t)$ with strike K and maturity T is:

$$V_c(t_0, B(t_0); K, T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T)}\max(B(T) - K, 0)\middle|\mathcal{F}(t_0)\right],$$

- Calculation of $V_c(t_0, B(t_0); K, T)$ suffers from the curse of dimensionality – can we overcome it?
- What if the options are American or Asian or...?

Insights & Inspirations

- Dupire (1994), Dupire (1996), Gyöngy (1986): local volatility, mimicking marginal distributions of complex, multi-dimensional processes;
- Piterbarg (2006): Markovian Projection;
- Borovkova et al. (2012), Lee and Wang (2012): displaced lognormal volatility skews;
- Piterbarg (2005): ”effective” parameters;
- Brigo et al. (2003): moment-matching technique.

Idea in a nutshell

Goal: build a 1D local vol process: $\mathrm{d}\bar{B}(t) = r\bar{B}(t)\mathrm{d}t + \sigma_{LV}(t, \bar{B}(t))\bar{B}(t)\mathrm{d}W(t)$, which will – by design – produce the same European option prices as $B(t)$, but at considerably less computational effort.

But how to get $\sigma_{LV}(t, \bar{B}(t))$?

- Map the multi-dimensional basket onto a collection of marginal distributions generated by ”simpler” processes;
- Use the calibrated marginals to generate European option prices on the basket;
- Use option prices to generate LV surface $\rightarrow \sigma_{LV}(t, \bar{B}(t))$;
- Use LV model to price path-dependent and exotic derivatives leveraging the one-dimensional representation.

Dupire/Gyöngy recap

When will $\mathrm{d}\bar{B}(t) = r\bar{B}(t)\mathrm{d}t + \sigma_{LV}(t, \bar{B}(t))\bar{B}(t)\mathrm{d}W(t)$, produce the same option prices as $B(t)$?

- repricing of European options between 2 models will be ensured iff they generate the same marginals distributions at any given time point;
- $\sigma^2_{LV}(T, K)$ has the interpretation of the conditional expectation of the stochastic variance of $B(t)$;
- $\sigma^2_{LV}(T, K)$ is given by the prices (equivalently, implied vols) of basket call/put options for a range of strikes, K_i, and maturities, T_j through the following formula:

$$\sigma^2_{LV}(T_j, K_i) = \frac{\dfrac{\partial V_c(t_0, B(t_0); K_i, T_j)}{\partial T} + rK_i \dfrac{\partial V_c(t_0, B(t_0); K_i, T_j)}{\partial K_i}}{\dfrac{1}{2} K_i^2 \dfrac{\partial^2 V_c(t_0, B(t_0); K_i, T_j)}{\partial K_i^2}},$$

Effective Local Volatility: Model Specification

For all T_j approximate $V_c(\cdot)$ via a projection of the underlying basket on a one-dimensional process Y_j, s.t.,

$$\begin{aligned} V_c(t_0, B(t_0); K, T_j) &\approx \hat{V}_c(t_0, Y_j(t_0); K, T_j) \\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_j)} \max(Y_j(T_j) - K, 0) \middle| \mathcal{F}(t_0)\right], \end{aligned}$$

with

$$\mathrm{d}Y_j(t) = rY_j(t)\mathrm{d}t + \xi_j Y_j(t)\mathrm{d}W(t),$$

where we impose a condition that at every T_j the process $Y_j(t)$ satisfies:

$$\forall T_j : \quad \min_{\xi_j} ||Y_j(T_j) - B(T_j)||_{L^p}.$$

Mapping the basket $B(\cdot)$ on processes $Y_i(t)$

For each expiry date T_j we will have one corresponding process $Y_j(t)$ that will be calibrated by mapping the basket $B(T_j)$.

Table:

method	T_1	T_2	T_3	...	T_N
$B(t)$	$B(T_1)$	$B(T_2)$	$B(T_3)$	...	$B(T_N)$
$Y_1(t)$	$\boxed{Y_1(T_1)}$			...	
$Y_2(t)$	$Y_2(T_1)$	$\boxed{Y_2(T_2)}$		...	
$Y_3(t)$	$Y_3(T_1)$	$Y_3(T_2)$	$\boxed{Y_3(T_3)}$	...	
...	...	...	...	...	
$Y_N(t)$	$Y_N(T_1)$	$Y_N(T_2)$	$Y_N(T_3)$	...	$\boxed{Y_N(T_N)}$

- Parameters of each $Y_j(t)$ select such that its first three moments match the corresponding moments of the basket.

Case 1: Lognormal dynamics of basket constituents

Assume that:

$$\forall i \ \ \mathrm{d}S_i(t) = rS_i(t)\mathrm{d}t + \sigma_i S_i(t)\mathrm{d}W_i(t), \quad \mathrm{d}W_i(t)\mathrm{d}W_j(t) = \rho_{i,j}\mathrm{d}t.$$

- Unfortunately, the problem of the distribution of the sum of lognormals remains unresolved...
- ...but we can prove the following

Proposition (Implied Volatility Skew for a Basket)

Implied volatility for the basket $B(t)$ is increasing in strike K, i.e.:

$$\frac{\partial \sigma_B}{\partial K} > 0.$$

- This suggests the projection of $B(t)$ on a displaced diffusion process which also generates a skew.

Displaced Diffusion recap

- A classical displaced diffusion (DD) process $S_d(t)$ is defined as a displacement of a lognormal process $S(t)$ with parameter $\theta \in \mathbb{R}$

$$S_d(t) = S(t) + \theta, \quad \mathrm{d}S(t) = \sigma_d S(t)\mathrm{d}W(t),$$

with the following dynamics for $S_d(t)$

$$\mathrm{d}S_d(t) = \sigma_d \left(S_d(t) - \theta\right) \mathrm{d}W(t), \quad S_d(t_0) = S(t_0) + \theta.$$

- As shown by Lee et al. implied vols for DD are bounded and monotonic:

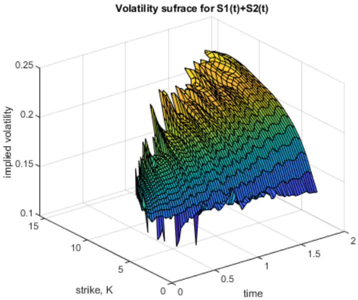
$$\text{sgn}\ \frac{\partial \sigma_{imp}(K,T)}{\partial K} = \text{sgn}\ \theta,$$

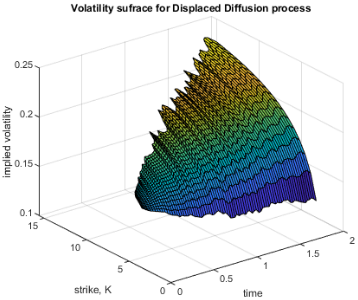
and for $\theta > 0$: $\sigma_{imp} < \sigma$ and for $\theta < 0$: $\sigma_{imp} > \sigma$ for $T > 0$.

- Moreover, the asymptotic implied volatilities for $T \to 0$ are known explicitly:

$$\lim_{T\to 0} \sigma_{imp}(K,T) = \begin{cases} \frac{\sigma \log(S(t_0)/K}{\log((S(t_0)-\theta)/(K-\theta))} & \text{for} \quad K \neq S(t_0) \\ \sigma(1-\theta/S(t_0)) & \text{for} \quad K = S(t_0). \end{cases}$$

Implied volatility surface for a basket vs. a Displaced Diffusion





Matching moments between $B(t)$ and DD processes

For lognormal diffusions it can be shown that

$$
\begin{aligned}
\mathbb{E}[B(t)] &= \sum_{i=1}^{N} \omega_i S_i(t_0) \\
\mathbb{E}[B^2(t)] &= \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i \omega_j S_i(t_0) S_j(t_0) \mathrm{e}^{\sigma_i \sigma_j \rho_{i,j} t} \\
\mathbb{E}[B^3(t)] &= \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \omega_i \omega_j \omega_k S_i(t_0) S_j(t_0) S_k(t_0) \mathrm{e}^{\sigma_i \sigma_j \rho_{i,j} t + \sigma_i \sigma_k \rho_{i,k} t + \sigma_j \sigma_k \rho_{j,k} t}.
\end{aligned}
$$

For each T_j Set $Y_j(T_j) := S(T_j) + \theta_j$ where:

$$
\mathrm{d}S(t) = \boxed{\sigma_j} S(t) \mathrm{d}W(t) \text{ and } S(t_0) := \sum_{i=1}^{N} \omega_i S_i(t_0) - \boxed{\theta_j}.
$$

By construction,

$$
\mathrm{E}[\mathrm{Y}_j(T_j)] = S(t_0) + \theta_j = \sum_{i=1}^{N} \omega_i S_i(t_0) = \mathbb{E}[B(T_j)].
$$

Matching moments between $B(t)$ and DD processes

To match the higher moments we prove the following:

Proposition (Parameters σ_j and θ_j)

Optimal parameters that minimize $\min_{\theta,\sigma} \quad \sum_{i=2}^{3} ||\mathbb{E}[B^i(t)] - \mathbb{E}[Y_j^i(t)]||_{L^2}$. *are given by:*

$$\sigma^2 t = \log\left(\frac{m_2 - 2(m_1 - \theta)\theta - \theta^2}{(m_1 - \theta)^2}\right),$$

$$0 = a_1\theta^3 + a_2\theta^2 + a_3\theta + a_4,$$

with

$$a_1 = 2m_1^3 - 3m_2 m_1 + m_3,$$

$$a_2 = -3m_1^4 + 3_1^2 m_2 - 3m_3 m_1 + 3m_2^2,$$

$$a_3 = 3m_1^3 m_2 + 3m_3 m_1^2 - 6m_1 m_2^2,$$

$$a_4 = -m_3 m_1^3 + m_2^3.$$

and where $m_1 := \mathbb{E}[B(t)]$, $m_2 := \mathbb{E}[B^2(t)]$ and $m_3 := \mathbb{E}[B^3(t)]$.

Case 2: Basket of stocks under the Heston (1993) model

Assume that each individual stock follows the Heston model:

$$\begin{aligned} \mathrm{d}S_j(t) &= rS_j(t)\mathrm{d}t + \sqrt{v_j(t)}S_j(t)\mathrm{d}W_{j,1}(t), \quad S_j(t_0) > 0, \\ \mathrm{d}v_j(t) &= \kappa_j(\bar{v}_j - v_j(t))\mathrm{d}t + \gamma_j\sqrt{v_j(t)}\mathrm{d}W_{j,2}(t), \quad v_j(t_0) > 0, \end{aligned}$$

with correlations

$$\begin{aligned} \mathrm{d}W_{j,1}(t)\mathrm{d}W_{j,2}(t) &= \rho_j\mathrm{d}t \\ dW_{j,1}(t)\mathrm{d}W_{k,1}(t) &= \rho_{j,k}\mathrm{d}t \ \text{ and} \\ dW_{j,2}(t)\mathrm{d}W_{k,2}(t) &= 0\cdot\mathrm{d}t. \end{aligned}$$

Characteristic function for the Heston model

$$\mathbb{E}[S_j^n(t)] = \mathbb{E}[\mathrm{e}^{n \log S_j(t)}] = \phi_{\log S_j(t)}(-in), \quad i \in \mathbb{C}, \quad \phi_{\log S_j(t)}(u) = \mathbb{E}[\mathrm{e}^{iu \log S_j(t)}],$$

where $\phi_{\log S_j(t)}(u)$ for the Heston model is given by:

Definition

The ChF for the Heston model is given by:

$$\phi_{\log S_j(T)}(u; t_0, t) = \exp\left(iu \log(S_j(t_0)) + \bar{C}_j(u, t - t_0)v(t_0) + \bar{A}_j(u, t - t_0)\right),$$

with complex values functions $A_j(u, t - t_0)$ and $C_j(u, t - t_0)$ given by:

$$\bar{C}_j(u,\tau) = \frac{1 - \mathrm{e}^{-D_{1,j}(t-t_0)}}{\gamma_j^2(1 - g_j \mathrm{e}^{-D_{1,j}\tau})}\left(\kappa_j - \gamma_j \rho_j iu - D_{1,j}\right),$$

$$\bar{A}_j(u,\tau) = r(iu - 1)\tau + \frac{\kappa_j \bar{v}_j (t - t_0)}{\gamma_j^2}\left(\kappa_j - \gamma_j \rho_j iu - D_{1,j}\right) - \frac{2\kappa_j \bar{v}_j}{\gamma_j^2} \log\left(\frac{1 - g_j \mathrm{e}^{-D_{1,j}\tau}}{1 - g_j}\right),$$

for $\tau = t - t_0$ and $D_{1,j} = \sqrt{(\kappa_j - \gamma_j \rho_j iu)^2 + (u^2 + iu)\gamma_j^2}$ and $g_j = \dfrac{\kappa_j - \gamma_j \rho_j iu - D_{1,j}}{\kappa_j - \gamma_j \rho_j iu + D_{1,j}}$.

Moments for the Basket under the Heston model

$$\mathbb{E}[B^p(t)] = \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_p=1}^N \omega_{i_1} \dots \omega_{i_p} \mathbb{E}[S_{i_1}(t) S_{i_2}(t) \dots S_{i_p}(t)],$$

In particular for the first three moments we have:

$$\begin{aligned} \mathbb{E}[B(t)] &= \sum_{i=1}^N \omega_i F_i(t_0), \quad F_i(t_0) = S_i(t_0) \mathrm{e}^{rt} \\ \mathbb{E}[B^2(t)] &= \sum_{i=1}^N \sum_{j=1}^N \omega_i \omega_j \Big(\rho_{i,j} \sigma_i(t) \sigma_j(t) + \mathbb{E}[S_i(t)] \mathbb{E}[S_j(t)] \Big), \end{aligned}$$

where $\sigma_i^2(t) = \mathbb{E}[S_i^2(t)] - \mathbb{E}^2[S_i(t)]$, with $\mathbb{E}[S_i^2(t)]$ and $\mathbb{E}[S_j^2(t)]$ defined above.
For the third moment we find:

$$\mathbb{E}[B^3(t)] = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \omega_i \omega_j \omega_k \mathbb{E}[S_i(t) S_j(t) S_k(t)],$$

Moments for the Basket under the Heston model

The third moment, alternatively, can be written as:

$$\begin{aligned}\mathbb{E}[B^3(t)] &= \sum_{i=1}^{N}\omega_i\mathbb{E}[S_i^3(t)] + 3\sum_{i=1}^{N}\sum_{j=1}^{i-1}\omega_i^2\omega_j \boxed{\mathbb{E}[S_i^2(t)S_j(t)]} + 3\sum_{i=1}^{N}\sum_{j=1}^{i-1}\omega_i\omega_j^2 \boxed{\mathbb{E}[S_i(t)S_j^2(t)]} \\ &+ 6\sum_{i=1}^{N}\sum_{j=1}^{i-1}\sum_{k=1}^{j-1}\omega_i\omega_j\omega_k \boxed{\mathbb{E}[S_i(t)S_j(t)S_k(t)]}.\end{aligned}$$

For the cross expectations we perform a projection on lognormal process, i.e.,

$$S_i(t) \approx \bar{S}_i(t) = F_i(t_0)\exp\left(-\frac{1}{2}\sigma_i^2 t + \sigma_i\bar{W}_i(t)\right), \quad \sigma_i = \sqrt{\frac{1}{t}\log\left(\frac{\mathbb{E}[S_i^2(t)]}{F_i^2(t_0)}\right)},$$

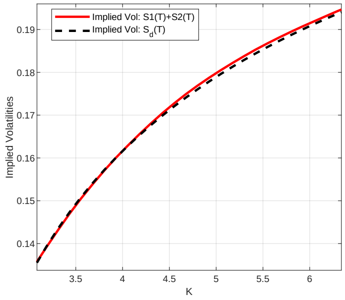
where σ_i is such that $\mathbb{E}[S_i^2(t)] = \mathbb{E}[\bar{S}_i^2(t)]$, and $\mathrm{d}\bar{W}_i(t)\mathrm{d}\bar{W}_j(t) = \rho_{i,j}\mathrm{d}t$. This yields:

$$\begin{aligned}\mathbb{E}[S_i(t)S_j(t)S_k(t)] &\approx F_i(t_0)F_j(t_0)F_k(t_0)\mathrm{e}^{\sigma_i\sigma_j\rho_{i,j}t+\sigma_i\sigma_k\rho_{i,k}t+\sigma_j\sigma_k\rho_{j,k}t}, \\ \mathbb{E}[S_i^2(t)S_j(t)] &\approx F_i^2(t_0)F_j(t_0)\mathrm{e}^{2\sigma_i\sigma_j\rho_{i,j}t+\sigma_i^2 t}.\end{aligned}$$

Example 1: basket of 2 stocks driven by GBM

Initial parameters

$S_1(t_0) = 1.5$, $S_2(t_0) = 2.5$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $\rho = -0.7$, $r = 0.01$ and maturity is set to $T = 2$. Calculations based on 10^5 Monte Carlo paths.



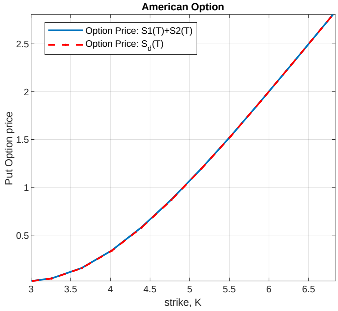
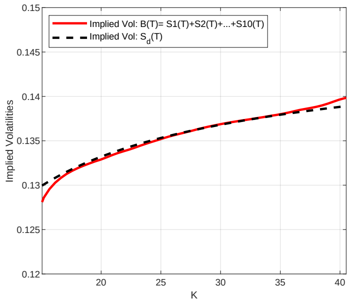


Figure: Left: Implied volatility surface for the basket $B(T)$ with $N = 2$ and for the displaced diffusion. Right: Option prices for American put for both models.

Example 2: basket of 10 stocks driven by GBM

Initial parameters

$\mathbf{S}(t_0) = [1.5,\ 2.0,\ 3.0,\ 1.2,\ 4.1,\ 5.2,\ 1.3,\ 2.4,\ 1.6,\ 2.4]$; and volatilities $\sigma = [0.1,\ 0.1,\ 0.2,\ 0.1,\ 0.2,\ 0.2,\ 0.14,\ 0.24,\ 0.3,\ 0.1]$; the correlation between all the underlying assets is set to $\rho = 0.1$; time-to-maturity is $T = 2$ and interest rate is set to $r = 0$.



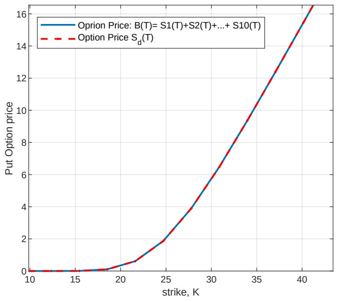


Figure: Left: Implied volatility surface for the basket $B(T)$ with $N = 10$ and for the displaced diffusion. Right: Option prices for American put for both models.

Example 3: basket of 5 stocks driven by Heston

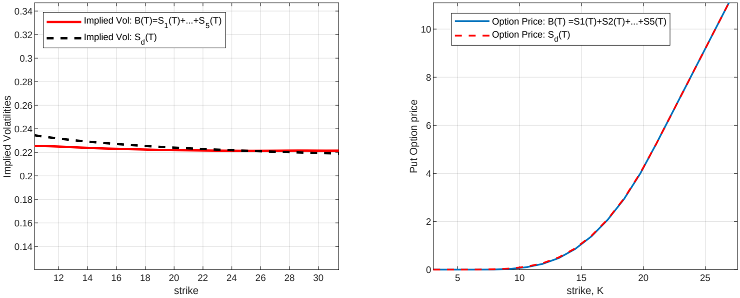


Figure: Left: Implied volatility surface for the basket $B(T)$ with $N = 5$ driven by the Heston model and for the displaced diffusion. Right: Option prices for American put for both models.

Example 4: basket of 100 stocks driven by GBM (á la FTSE 100)

Initial parameters

Prices and implied volatilities for $N = 100$ stocks based on UKX Index members as of Nov 15. The correlation between all the underlying assets is set to $\rho = 0.69$ (average index correlation); time-to-maturity is $T = 2$ and interest rate is set to $r = 0.02$.

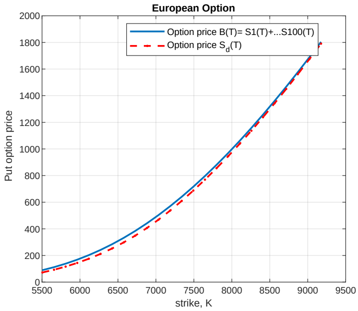


Figure: Left: European put option prices for the basket $B(T)$ with $N = 100$ and for the displaced diffusion. Right: Option prices for American put for both models.

Why bother with dimension reduction?

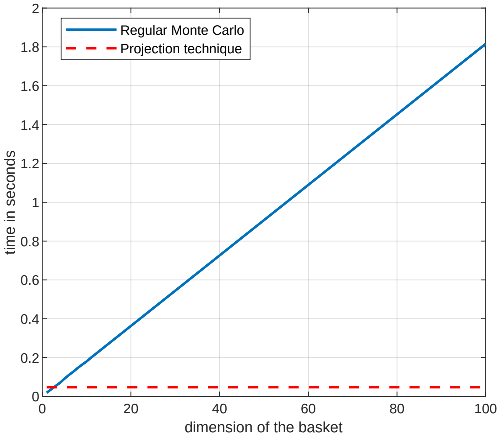


Figure: Timing results: Standard Monte Carlo (2000 paths) vs. the moment projection.

Wrap up & conclusions

- Novel way of using Markovian Projection: replacing a complex model with its simpler counterpart such that the two models agree on the prices of European options
- ”Effective” approach: no need to solve for or approximate conditional expectations directly, but rather determine the local volatility surface from option prices derived by matching marginal distributions.
- Matching marginals only gives good results for American options
- Dimension reduction = saving computation time without sacrificing much in terms of precision
- Still work in progress...
- Some loose ends to think about
  - More efficient/accurate moment matching?
  - Extension to stochastic local vol?
  - Calibration and extension to different payoffs